of the analytical approximate solution are examined and validated for the MDCWR. In addition, under the off-design conditions, the decreases in L/D and static margin are slight even though the angle of attack and Mach number alter. Moreover, after comparing the ECDWR with the present waverider, the advantage of longitudinal curvature is verified by an increased volumetric ratio, enhanced L/D, and wider range of flight conditions than ECDWR. Notably, the viscous drag will increase considerably, as a percentage of the total drag, as the generating angle for the waverider decreases and as the flight conditions result in turbulent flow.

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References

[1]Rasmussen, M. L., "Waverider Configurations Derived from Inclined Circular and Elliptic Cones," *Journal of Spacecraft and Rockets*, Vol. 17, No. 6, 1980, pp. 537–545.

[2]Rasmussen, M. L., Jischke, M. C., and Daniel, D. C., "Experimental Surface Pressure on Cone-Derived Waveriders for $M = 3$–5," *Journal of Spacecraft and Rockets*, Vol. 20, No. 6, 1983, pp. 539–545.

[3]Lin, S. C., and Rasmussen, M. L., "Cone-Derived Waverider with Combined Transverse and Longitudinal Curvature," AIAA Paper 88-0371, Jan. 1988.

[4]Lin, S. C., and Luo, Y. S., "Integrated Inlets for Cone-Derived Waverider with Combined Transverse and Longitudinal Curvature," *Journal of Spacecraft and Rockets*, Vol. 31, No. 4, 1994, pp. 609–614.

[5]Liao, J. R., Isacc, K. M., Miles, J. B., and Tsai, B. J., "Navier–Stokes Simulation for Cone-Derived Waverider," *AIAA Journal*, Vol. 30, No. 6, 1992, pp. 1521–1528.

[6]He, X., and Rasmussen, M. L., "Computational Analysis of Off-Design Waveriders," *Journal of Spacecraft and Rockets*, Vol. 31, No. 2, 1994, pp. 345–353.

[7]Roe, P. L., "Approximate Riemann Solvers, Parameter Vectors and Difference Schemes," *Journal of Computational Physics*, Vol. 43, No. 2, 1981, pp. 357–372.

[8]Thompson, J. F., and Warsi, Z. U. A., "Boundary-Fitted Coordinate Systems for Numerical Solution of Partial Differential Equations: A Review," *Journal of Computational Physics*, Vol. 47, No. 1, 1982, pp. 101–108.

[9]Van Driest, E. R., "Investigation of Laminar Boundary Layer in Compressible Fluids Using the Crocco Method," NACA TN 2579, Jan. 1952.

Momentum and Vortex Theory of Rotor Blade Wakes

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Introduction

THE customary impulse-momentum theory for a rotor disk in vertical flight establishes a relation between thrust and induced velocity in the far wake. Steady frictionless flow is assumed, and uniform ambient pressure in the far wake, corresponding to infinitely many blades. The exact flow in the wake due to a finite number of blades can be calculated using vortex theory. The customary approximation is to assume a rigid vortex sheet, which flows down with uniform wake velocity for the ideal rotor operation.

Prandtl[1,2] has used the approximation that the blade lift from momentum theory only differs substantially at the blade tips. He approximated the flow from the blade tips in the far wake by a two-dimensional flow of parallel half-infinite vortex sheets. This model is a good approximation of the exact calculations of the rigid vortex model by Goldstein.[3] Goldstein's model has led Theodorsen[4] to conclude that the pressure in the far wake is higher than the surrounding atmospheric pressure, contrary to impulse-momentum theory. More recently Bramwell[5–7] has used the two-dimensional tip vortex sheet approximation of Prandtl to calculate the overpressure $(p - p_\infty)/\rho = \frac{1}{2}V_\infty^2$ in the far wake, where V_∞ is the uniform flow velocity outside upwards, relative to the vortex sheet cascade, and the surrounding atmospheric pressure is p_∞. This contradicts the assumption and conclusion of the impulse-momentum theory, albeit by a small amount. In a recent revision of Theodorsen's theory, Ribner[8] still uses this result.

However, this theory cannot be correct. Betz[9] has pointed out that horizontal finite forces at the vortex sheet tips would be required to sustain the rigid vortex sheet, and the absence of these forces is the cause for roll-up.

It will be shown subsequently that, to correct the unbalance from the rigid state, the very edge of the vortex sheet, with infinitesimal vortex strength, moves upwards with infinite velocity, therefore creating an unsteady flow with a corresponding additional pressure field. This concept was presented originally in Ref. 10.

Vortex Sheet Cascade

As approximation the two-dimensional flow over a cascade of finite width sheets is used. The potential flow is obtained by conformal mappings like those given by Kober[11] or Betz.[12] The width of the cascade is taken as 2, the right edge being at $x = 1$, and the pitch as h. The loading of the elliptic wing is approached as $h \to \infty$ with constant downwash velocity V_∞, Prandtl's tip wake flow is approached as $h \to 0$ with constant product $V_\infty h$, and uniform rectangular loading is approached as $h \to 0$ with constant V_∞.

Steady Flow

The pressure of steady flow relative to the vortex sheet from Bernoulli's equation is shown as carpet plot along lines parallel to the vortex sheets in Fig. 1 for $h = 0.5$. Practical values of h are of this order. As an example, the Westland model 30 helicopter at max takeoff weight of 12,800 lb would produce a far wake at hover with spacing of $h = 0.37$. The pressure is seen to be almost $\frac{1}{2}\rho V_\infty^2$ between the sheets except for end effects.

From Blasius' theorem for steady flow horizontal forces at the edges, $f_x = c_1 \frac{1}{2}\rho V_\infty^2$, are found, where c_1 is a constant depending on h. Although these forces would not enter the vertical balance in momentum theory, they do require the overpressure between the sheets to satisfy the horizontal equilibrium of half a vortex sheet.

Unsteady Flow

In this simplified model the unsteady flow starts when the edge forces of steady flow are removed instantaneously. Therefore all of the edge vortices of infinitely small size start suddenly with infinite

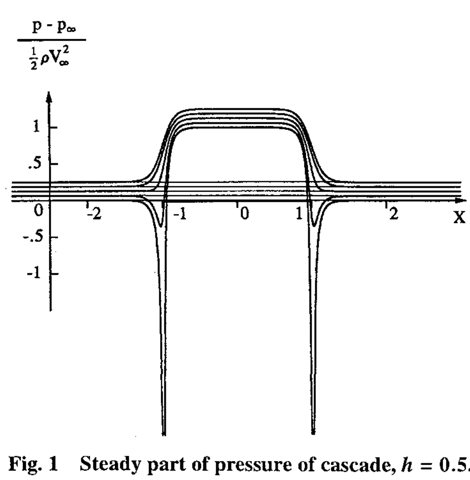

Fig. 1 Steady part of pressure of cascade, h = 0.5.

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velocity upwards, such that no force results from the Kutta lift theorem. This is the beginning of the roll-up of the vortex sheet. The roll-up is therefore not a matter of stability but a matter of equilibrium.

The vorticity on the edge of the sheet as $x \to 1$ is found as $\gamma = 2c_2 V_\infty/\sqrt{(1-x)}$; therefore, the infinitesimally small edge vortices are assumed to have strength proportional to $\mathrm{d}x$, $\mathrm{d}k = c_3\gamma\,\mathrm{d}x$, which becomes $\mathrm{d}k = c_3 V_\infty\sqrt{(\mathrm{d}x)}$ as the term $(1-x) \to \mathrm{d}x$. The vertical velocity just outside the very edge of the sheet as $x \to 1$ is found as $v = c_4 V_\infty/\sqrt{(x-1)}$. Therefore, the infinite mean edge-vortex velocity, which is half of the velocity at the outside of the edge, is $V_k = c_4 V_\infty/\sqrt{(\mathrm{d}x)}$, also as the term $(x-1) \to \mathrm{d}x$. The constants c_2, c_3, and c_4 depend on the cascade pitch h. Therefore the product $\mathrm{d}k\,V_k$ is a finite quantity. Its value can be calculated by equating the force from Blasius' theorem to the force from the Kutta lift theorem, $f_x = \rho V_k\,\mathrm{d}k$, with the result

$$\mathrm{d}k\,V_k = \tfrac{1}{2}(h/a)V_\infty^2 \tag{1}$$

where the constant $a = (e^{2\pi/h}+1)/(e^{2\pi/h}-1)$.

Because there is no physical force available to hold the edge vortex in the rigid position, it swims upwards with the infinite velocity V_k, creating the unsteady flow of vortex sheets in dynamic equilibrium. The term $\partial\phi/\mathrm{d}t$ in Kelvin's equation is proportional to the product of the infinitesimal moving vortex strength and the infinite velocity V_k. (The partial differential is written in a modern form according to Ref. 13.) As a result, an unsteady part of the pressure proportional to the finite product $\mathrm{d}k\,V_k$ is contributed by the moving edge vortices.

The unsteady part of the edge force is computed from the unsteady part of Blasius' equation (see Ref. 14) and turns out to be equal but opposite to the steady edge force. Therefore, the total edge force in unsteady flow is zero as required by equilibrium.

The unsteady part of pressure from Kelvin's equation is shown as carpet plots along lines parallel to the sheet in Fig. 2. For small pitch h it is also constant between the sheets except for end effects and equal but opposite to the steady part. The resulting unsteady pressure at sections parallel to the x axis is shown in Fig. 3. At small pitch h this tends to constant ambient atmospheric pressure inside and outside the cascade, except for the infinite end effects at the edge vortices. The blade loading from the vortex sheet theory is not affected by the unsteady motion.

It can be shown that the infinite edge vortex velocity $V_k \to \frac{1}{2}V_\infty$ and that the edge vortex density $\mathrm{d}k/\mathrm{d}y \to V_\infty$ as the cascade pitch $h \to 0$ for infinitely many blades. This corresponds with the correct relative velocity and vortex density of the vortex walls in equilibrium that separate the uniform wake of a rotordisk from the ambient atmosphere. The velocity $\frac{1}{2}V_\infty$ also corresponds with the result given by Betz[9] for the approximate velocity of tip vortices of the propeller wake.

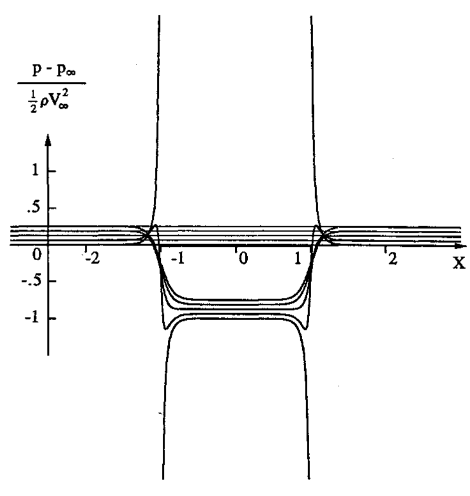

Fig. 2 Unsteady part of pressure of cascade, h = 0.5.

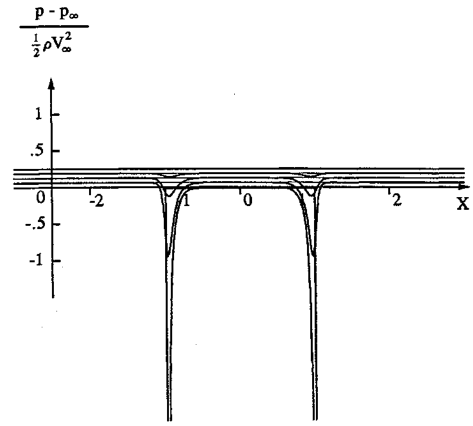

Fig. 3 Unsteady pressure of cascade, h = 0.5.

Approximation by Discrete Vortices

The theory of the unsteady motion of the cascade of vortex sheets is tested numerically with an infinite cascade of discrete vortex rows, along the same lines as by Moore,[15] but the edge vortex strengths were adjusted to obtain a uniform induced velocity at the inside vortices. Steady pressure and unsteady pressure from Bernoulli's and Kelvin's equations were calculated. The results were in excellent agreement with the theoretical for as few as 10 vortices.

Conclusions

The result of the unsteady theory is that the pressure in the far wake is equal to the ambient pressure, which is the same as the assumption of customary impulse-momentum theory. Therefore, the theories of impulse momentum and vortex sheets do correspond if the unsteady motion of the rolling up of the vortex sheets is not neglected, as this would constitute incorrect forces on the wake.

There is no reason to doubt the overpressure measurements in the wake reported by Bramwell,[5,7] but according to the preceding theory the explanation for that must be searched outside frictionless flow of the infinite wake.

References

[1]Prandtl, L., "Zusatz zu A. Betz: Schrauben-Propeller mit geringstem Energieverlust," *Nachrichten der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse*, Göttingen, 1919, pp. 213–217.

[2]Prandtl, L., "Applications of Modern Hydrodynamics to Aeronautics," NACA TR 116, 1921, pp. 7–61.

[3]Goldstein, S., "The Vortex Theory of Propellers," *Proceedings of the Royal Society of London*, Vol. 123, No. 792, 1929, pp. 440–465.

[4]Theodorsen, T., *Theory of Propellers*, McGraw–Hill, New York, 1948.

[5]Bramwell, A. R. S., "A Note on the Static Pressure in the Wake of a Hovering Helicopter Rotor," City Univ. Research Memorandum, Dept. of Aeronautics, Aero 73/3, London, Aug. 1973.

[6]Bramwell, A. R. S., *Helicopter Dynamics*, Edward Arnold, London, 1976.

[7]Bramwell, A. R. S., "On the Static Pressure in the Wake of a Hovering Rotor," *Vertica*, Vol. 1, No. 3, 1977, pp. 223–230.

[8]Ribner, H. S., "Ideal Efficiency of Propellers: Theodorsen Revisited," *Journal of Aircraft*, Vol. 27, No. 9, 1990, pp. 810–819.

[9]Betz, A., "Verhalten von Wirbelsystemen," *Zeitschrift für Angewandte Mathemathik und Mechanik*, Vol. 12, No. 3, 1932, pp. 164–174; see also "Behaviour of Vortex Systems," NACA TM 713, June 1933.

[10]Hassenpflug, W. C., "On the Correspondence between Momentum and Vortex Theories in Rotor Vertical Flight," *First South African Aeronautical Engineering Conference*, South African Inst. of Aeronautical Engineers, Pretoria, South Africa, May 1988, pp. 25–27.

[11]Kober, H., *Dictionary of Conformal Mappings*, Dover, New York, 1957.

[12]Betz, A., *Konforme Abbildung*, Springer, Berlin, 1964.

[13]Hassenpflug, W. C., "Revised Notation for Partial Derivatives," *Computers & Mathematics with Applications*, Vol. 26, No. 3, 1993, pp. 95–106.

[14]Milne-Thomson, L. M., *Theoretical Hydrodynamics*, 5th ed., Macmillan, London, 1968.

[15]Moore, D. W., "A Numerical Study of the Roll-Up of a Finite Vortex Sheet," *Journal of Fluid Mechanics*, Vol. 63, April 1974, pp. 225–235.